On the Hungarian matching schemes for secondary and higher education

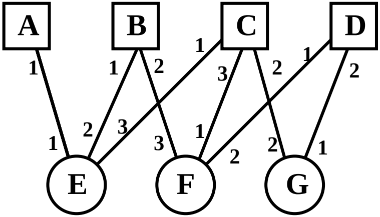
Péter Biró
Department of Computing Science
University of Glasgow
pbiro@dcs.gla.ac.uk
Joint work with Tamás Fleiner, Rob Irving and David Manlove

Keszthely
12 August 2008

University of Glasgow

Stable Marriage problem

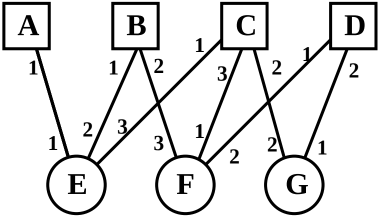
Gale, Shapley [1962]:
"College admission and the stability of marriage"



"Each person ranks those of the opposite sex in accordance with his or her preferences for a marriage partner."

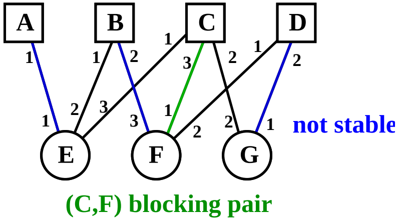
Stable Marriage problem

Gale, Shapley [1962]:
"College admission and the stability of marriage"



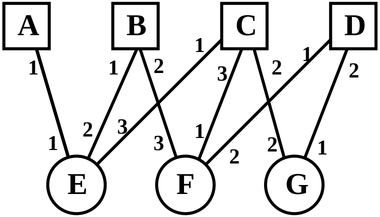
"Each person ranks those of the opposite sex in accordance with his or her preferences for a marriage partner."

A set of marriages is stable, if there is no "blocking pair": a man and a woman who are not married to each other but prefer each other to their actual mates.



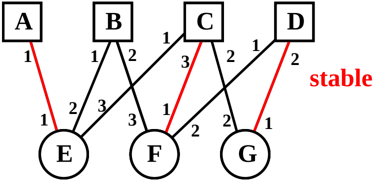
Stable Marriage problem

Gale, Shapley [1962]:
"College admission and the stability of marriage"



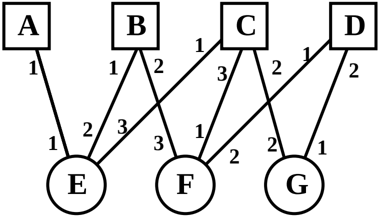
"Each person ranks those of the opposite sex in accordance with his or her preferences for a marriage partner."

A set of marriages is stable, if there is no "blocking pair": a man and a woman who are not married to each other but prefer each other to their actual mates.



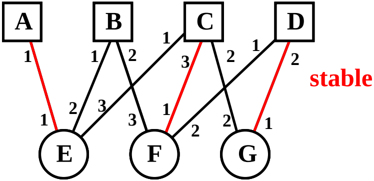
Stable Marriage problem

Gale, Shapley [1962]:
"College admission and the stability of marriage"



"Each person ranks those of the opposite sex in accordance with his or her preferences for a marriage partner."

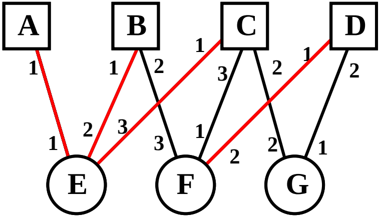
A set of marriages is stable, if there is no "blocking pair": a man and a woman who are not married to each other but prefer each other to their actual mates.



Gale-Shapley 1962: The deferred-acceptance algorithm finds a stable matching in $O(m)$ time. This matching is *man-optimal*.

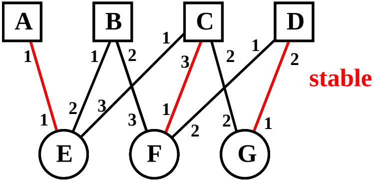
Stable Marriage problem

Gale, Shapley [1962]:
“College admission and the stability of marriage”



“Each person ranks those of the opposite sex in accordance with his or her preferences for a marriage partner.”

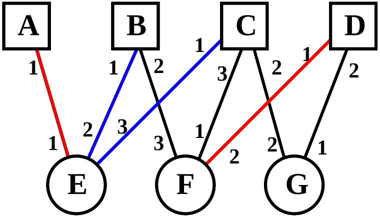
A set of marriages is stable, if there is no “blocking pair”: a man and a woman who are not married to each other but prefer each other to their actual mates.



Gale-Shapley 1962: The deferred-acceptance algorithm finds a stable matching in $O(m)$ time. This matching is *man-optimal*.

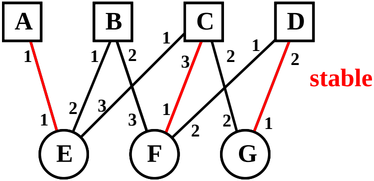
Stable Marriage problem

Gale, Shapley [1962]:
"College admission and the stability of marriage"



"Each person ranks those of the opposite sex in accordance with his or her preferences for a marriage partner."

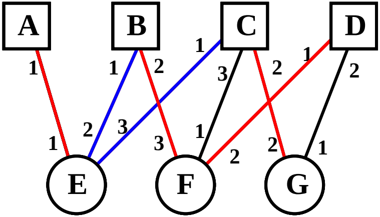
A set of marriages is stable, if there is no "blocking pair": a man and a woman who are not married to each other but prefer each other to their actual mates.



Gale-Shapley 1962: The deferred-acceptance algorithm finds a stable matching in $O(m)$ time. This matching is *man-optimal*.

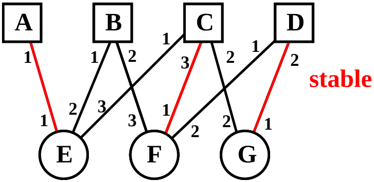
Stable Marriage problem

Gale, Shapley [1962]:
“College admission and the stability of marriage”



“Each person ranks those of the opposite sex in accordance with his or her preferences for a marriage partner.”

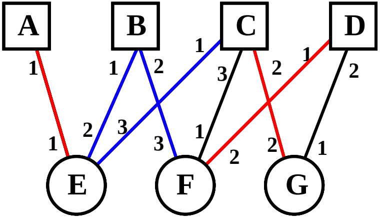
A set of marriages is stable, if there is no “blocking pair”: a man and a woman who are not married to each other but prefer each other to their actual mates.



Gale-Shapley 1962: The deferred-acceptance algorithm finds a stable matching in $O(m)$ time. This matching is *man-optimal*.

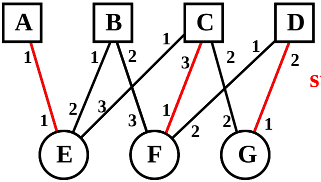
Stable Marriage problem

Gale, Shapley [1962]:
"College admission and the stability of marriage"



"Each person ranks those of the opposite sex in accordance with his or her preferences for a marriage partner."

A set of marriages is stable, if there is no "blocking pair": a man and a woman who are not married to each other but prefer each other to their actual mates.



stable

Gale-Shapley 1962: The deferred-acceptance algorithm finds a stable matching in $O(m)$ time. This matching is *man-optimal*.

Stable Marriage problem $\longrightarrow$ College Admissions

Each college C has a *quota* q_C.

(many-to-one) matching:
No college can admit more applicants than its quota.

Stable Marriage problem $\longrightarrow$ College Admissions

Each college C has a *quota* q_C.

(many-to-one) matching:
No college can admit more applicants than its quota.

The stability of a matching:
If an applicant a_i is not matched to a college C, then
– either a_i is matched to a better college
– or C has filled its quota with better applicants than a_i

Stable Marriage problem $\longrightarrow$ College Admissions

Each college C has a *quota* q_C.

(many-to-one) matching:
No college can admit more applicants than its quota.

The stability of a matching:
If an applicant a_i is not matched to a college C, then
– either a_i is matched to a better college
– or C has filled its quota with better applicants than a_i

college-proposing algorithm:
– each college C proposes to its q_C favorite applicants
– each applicant who receives more than one offer rejects all but her favorite...

Stable Marriage problem ⟶ College Admissions

Each college C has a *quota* q_C.

(many-to-one) matching:
No college can admit more applicants than its quota.

The stability of a matching:
If an applicant a_i is not matched to a college C, then
– either a_i is matched to a better college
– or C has filled its quota with better applicants than a_i

college-proposing algorithm:
– each college C proposes to its q_C favorite applicants
– each applicant who receives more than one offer rejects all but her favorite...

applicant-proposing algorithm:
– each applicant proposes to her favorite college
– each college C who receives more than q_C proposals rejects all but its q_C favorite applicants...

The Gale–Shapley algorithm in practice

Allocating residents to positions:

- National Resident Matching Program since 1952!
- and many other professions in the US and other countries...

The Gale–Shapley algorithm in practice

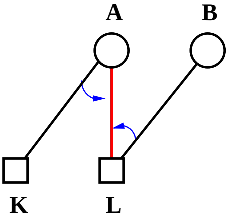
Allocating residents to positions:

- National Resident Matching Program since 1952!
- and many other professions in the US and other countries...

Admission systems in education:

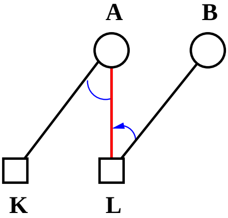
- New York high schools since 2004,
  Boston high schools since 2005
- Higher education admissions in Spain (1998)
- Higher education admissions in Hungary since 1985
- Secondary school admissions in Hungary since 2000
  (Original Gale–Shapley model and algorithm!)

A hard problem: Stable Marriage with ties



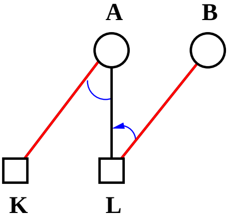
- In case of strict preferences, the size of the stable matchings and the set of matched agents are fixed.

A hard problem: Stable Marriage with ties



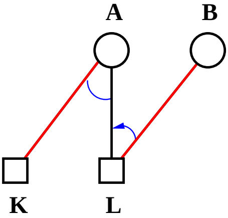
- In case of strict preferences, the size of the stable matchings and the set of matched agents are fixed.
- In case of ties, the size of the **weakly** stable matchings may differ.

A hard problem: Stable Marriage with ties



- In case of strict preferences, the size of the stable matchings and the set of matched agents are fixed.
- In case of ties, the size of the **weakly** stable matchings may differ.

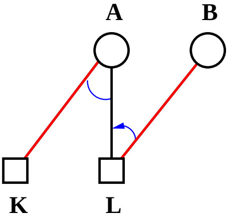
A hard problem: Stable Marriage with ties



- In case of strict preferences, the size of the stable matchings and the set of matched agents are fixed.
- In case of ties, the size of the **weakly** stable matchings may differ.

MAX SMTI: The problem of finding a maximum size weakly stable matching.

A hard problem: Stable Marriage with ties

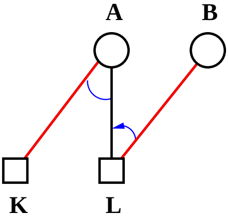


- In case of strict preferences, the size of the stable matchings and the set of matched agents are fixed.
- In case of ties, the size of the **weakly** stable matchings may differ.

MAX SMTI: The problem of finding a maximum size weakly stable matching.

Manlove *et al.* (2002): MAX SMTI is NP-hard.

A hard problem: Stable Marriage with ties



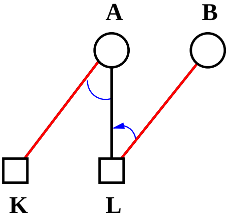
- In case of strict preferences, the size of the stable matchings and the set of matched agents are fixed.
- In case of ties, the size of the **weakly** stable matchings may differ.

MAX SMTI: The problem of finding a maximum size weakly stable matching.

Manlove *et al.* (2002): MAX SMTI is NP-hard.

Halldórsson *et al.* (2007): MAX SMTI is not approximable within $\frac{21}{19} - \varepsilon$ for any $\varepsilon > 0$, unless $P = NP$.

A hard problem: Stable Marriage with ties



- In case of strict preferences, the size of the stable matchings and the set of matched agents are fixed.
- In case of ties, the size of the **weakly** stable matchings may differ.

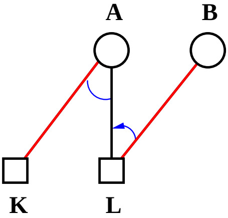
MAX SMTI: The problem of finding a maximum size weakly stable matching.

Manlove *et al.* (2002): MAX SMTI is NP-hard.

Halldórsson *et al.* (2007): MAX SMTI is not approximable within $\frac{21}{19} - \varepsilon$ for any $\varepsilon > 0$, unless $P = NP$.

Király (2008): Polynomial-time $\frac{3}{2}$-approximation.

A hard problem: Stable Marriage with ties



- In case of strict preferences, the size of the stable matchings and the set of matched agents are fixed.
- In case of ties, the size of the **weakly** stable matchings may differ.

MAX SMTI: The problem of finding a maximum size weakly stable matching.

Manlove *et al.* (2002): MAX SMTI is NP-hard.

Halldórsson *et al.* (2007): MAX SMTI is not approximable within $\frac{21}{19} - \varepsilon$ for any $\varepsilon > 0$, unless $P = NP$.

Király (2008): Polynomial-time $\frac{3}{2}$-approximation.

Application: Scottish Foundation Allocation Scheme (SFAS)
2006-2007: 781 residents, 53 hospitals, total capacity 789.
Maximum size weakly stable matching found was of size 744.

Higher education admissions in Hungary

Applicants has ranking lists over particular studies they apply for: study (field, type, state/privately financed), faculty, university

Adam's list			
1.	CS reg. state f. at BUTE (Budapest)		
2.	Maths reg. state f. at BUTE (Budapest)		
3.	Economics reg. state f. at Uni. of Pécs		
4.	CS reg. privately f. at BUTE (Budapest)		

Higher education admissions in Hungary

Applicants has ranking lists over particular studies they apply for: study (field, type, state/privately financed), faculty, university

Adam's list		score	
1.	CS reg. state f. at BUTE (Budapest)	117	
2.	Maths reg. state f. at BUTE (Budapest)	108	
3.	Economics reg. state f. at Uni. of Pécs	113	
4.	CS reg. privately f. at BUTE (Budapest)	117	

Applicants receive scores at each place they applied for.

Higher education admissions in Hungary

Applicants has ranking lists over particular studies they apply for: study (field, type, state/privately financed), faculty, university

	Adam's list	score	limit
1.	CS reg. state f. at BUTE (Budapest)	117	121
2.	Maths reg. state f. at BUTE (Budapest)	108	108
3.	Economics reg. state f. at Uni. of Pécs	113	114
4.	CS reg. privately f. at BUTE (Budapest)	117	110

Applicants receive scores at each place they applied for.

A centralised program computes the score-limits.

Higher education admissions in Hungary

Applicants has ranking lists over particular studies they apply for: study (field, type, state/privately financed), faculty, university

Adam's list		score	limit
1.	CS reg. state f. at BUTE (Budapest)	117	121
2.	**Maths reg. state f. at BUTE (Budapest)**	**108**	**108**
3.	Economics reg. state f. at Uni. of Pécs	113	114
4.	CS reg. privately f. at BUTE (Budapest)	117	110

Applicants receive scores at each place they applied for.

A centralised program computes the score-limits.

Each applicant is assigned to the **first place** on his list, where his score is greater than or equal to the limit.

1st special feature: TIES and stable score-limits

The applicants with the same score at a college are in tie, they are all accepted or rejected together!

A score-limit l is a mapping l : Colleges $\longrightarrow \mathbb{N}$ (actually ≤ 480).

1st special feature: TIES and stable score-limits

The applicants with the same score at a college are in tie, they are all accepted or rejected together!

A score-limit l is a mapping l : Colleges $\longrightarrow \mathbb{N}$ (actually ≤ 480).

Every score-limit l produces a particular matching $M(l)$.

1st special feature: TIES and stable score-limits

The applicants with the same score at a college are in tie, they are all accepted or rejected together!

A score-limit l is a mapping l : Colleges $\longrightarrow \mathbb{N}$ (actually ≤ 480).

Every score-limit l produces a particular matching $M(l)$.

A score-limit l is **feasible** if no quota is exceeded in $M(l)$.

1st special feature: TIES and stable score-limits

The applicants with the same score at a college are in tie, they are all accepted or rejected together!

A score-limit l is a mapping l : Colleges $\longrightarrow \mathbb{N}$ (actually ≤ 480).

Every score-limit l produces a particular matching $M(l)$.

A score-limit l is **feasible** if no quota is exceeded in $M(l)$.

A score-limit l is stable if it is feasible, and no college can decrease its limit without violating its quota (assuming that the others do not change their limits).

1st special feature: TIES and stable score-limits

The applicants with the same score at a college are in tie, they are all accepted or rejected together!

A score-limit l is a mapping l : Colleges $\longrightarrow \mathbb{N}$ (actually ≤ 480).

Every score-limit l produces a particular matching $M(l)$.

A score-limit l is **feasible** if no quota is exceeded in $M(l)$.

A score-limit l is stable if it is feasible, and no college can decrease its limit without violating its quota (assuming that the others do not change their limits).

Proposition: if no two applicants have the same scores at any college then l is stable $\iff$ $M(l)$ is stable in the original sense.

College-oriented algorithms until 2007

Studies:	Maths	CS	Economics
quotas:	≤ 50	≤ 500	≤ 500
25th appl.	Bill (135)	...	...
51st appl.	David (129)	...	...
85th appl.	Adam (125)	...	...
472th appl.		Adam (121)	...
473th appl.		David (120)	...
487th appl.		...	Bill (115)
488th appl.		...	Cliff (114)
501st appl.		...	Adam (114)
502nd appl.		Bill (120)	David (113)

Adam's list: Maths, Economics, CS
Bill's list: CS, Economics, Maths
Cliff's list: Economics
David's list: Maths, CS, Economics

College-oriented algorithms until 2007

Studies:	**Maths**	**CS**	**Economics**
quotas:	≤ 50	≤ 500	≤ 500
25th appl.	Bill (135)	...	...
51st appl.	David (129)	...	...
85th appl.	Adam (125)	...	...
472th appl.		Adam (121)	...
473th appl.		David (120)	...
487th appl.		...	Bill (115)
488th appl.		...	Cliff (114)
501st appl.		...	Adam (114)
502nd appl.		Bill (120)	David (113)

Adam's list: Maths, Economics, CS
Bill's list: CS, Economics, Maths
Cliff's list: Economics
David's list: Maths, CS, Economics

College-oriented algorithms until 2007

Studies:	**Maths**	**CS**	**Economics**
quotas:	≤ 50	≤ 500	≤ 500
25th appl.	Bill (135)	...	...
51st appl.	David (129)	...	...
85th appl.	Adam (125)	...	...
472th appl.		Adam (121)	...
473th appl.		David (120)	...
487th appl.		...	Bill (115)
488th appl.		...	Cliff (114)
501st appl.		...	Adam (114)
502nd appl.		Bill (120)	David (113)

Adam's list: Maths, Economics, CS
Bill's list: CS, Economics, Maths
Cliff's list: Economics
David's list: Maths, CS, Economics

College-oriented algorithms until 2007

Studies:	**Maths**	**CS**	**Economics**
quotas:	≤ 50	≤ 500	≤ 500
25th appl.	Bill (135)	...	...
51st appl.	David (129)	...	...
85th appl.	Adam (125)	...	...
472th appl.		Adam (121)	...
473th appl.		David (120)	...
487th appl.		...	Bill (115)
488th appl.		...	Cliff (114)
501st appl.		...	Adam (114)
502nd appl.		Bill (120)	David (113)

Adam's list: Maths, Economics, CS
Bill's list: CS, Economics, Maths
Cliff's list: Economics
David's list: Maths, CS, Economics

College-oriented algorithms until 2007

Studies:	**Maths**	**CS**	**Economics**
quotas:	≤ 50	≤ 500	≤ 500
25th appl.	Bill (135)	...	...
51st appl.	David (129)	...	...
85th appl.	Adam (125)	...	...
472th appl.		Adam (121)	...
473th appl.		David (120)	...
487th appl.		...	Bill (115)
488th appl.		...	Cliff (114)
501st appl.		...	Adam (114)
502nd appl.		Bill (120)	David (113)

Adam's list: Maths, Economics, CS
Bill's list: CS, Economics, Maths
Cliff's list: Economics
David's list: Maths, CS, Economics

Applicant-oriented algorithms since 2007

Studies:	**Maths**	**CS**	**Economics**
quotas:	≤ 50	≤ 500	≤ 500
25th appl.	Bill (135)	...	...
51st appl.	David (129)	...	...
85th appl.	Adam (125)	...	...
472th appl.		Adam (121)	...
473th appl.		David (120)	...
487th appl.		...	Bill (115)
488th appl.		...	Cliff (114)
501st appl.		...	Adam (114)
502nd appl.		Bill (120)	David (113)

Adam's list: Maths, Economics, CS
Bill's list: CS, Economics, Maths
Cliff's list: Economics
David's list: Maths, CS, Economics

Applicant-oriented algorithms since 2007

Studies:	**Maths**	**CS**	**Economics**
quotas:	≤ 50	≤ 500	≤ 500
25th appl.	Bill (135)	...	...
51st appl.	David (129)	...	...
85th appl.	Adam (125)	...	...
472th appl.		Adam (121)	...
473th appl.		David (120)	...
487th appl.		...	Bill (115)
488th appl.		...	Cliff (114)
501st appl.		...	Adam (114)
502nd appl.		Bill (120)	David (113)

Adam's list: Maths, Economics, CS
Bill's list: CS, Economics, Maths
Cliff's list: Economics
David's list: Maths, CS, Economics

Applicant-oriented algorithms since 2007

Studies:	**Maths**	**CS**	**Economics**
quotas:	≤ 50	≤ 500	≤ 500
25th appl.	Bill (135)	...	...
51st appl.	David (129)	...	...
85th appl.	Adam (125)	...	...
472th appl.		Adam (121)	...
473th appl.		David (120)	...
487th appl.		...	Bill (115)
488th appl.		...	Cliff (114)
501st appl.		...	Adam (114)
502nd appl.		Bill (120)	David (113)

Adam's list: Maths, Economics, CS
Bill's list: CS, Economics, Maths
Cliff's list: Economics
David's list: Maths, CS, Economics

Applicant-oriented algorithms since 2007

Studies:	**Maths**	**CS**	**Economics**
quotas:	$\leq$ 50	$\leq$ 500	$\leq$ 500
25th appl.	Bill (135)	...	...
51st appl.	David (129)	...	...
85th appl.	Adam (125)	...	...
472th appl.		Adam (121)	...
473th appl.		David (120)	...
487th appl.		...	Bill (115)
488th appl.		...	Cliff (114)
501st appl.		...	Adam (114)
502nd appl.		Bill (120)	David (113)

Adam's list: Maths, Economics, CS
Bill's list: CS, Economics, Maths
Cliff's list: Economics
David's list: Maths, CS, Economics

Generalisation of the Gale-Shapley theorem

Observation: The applicant/college-oriented algorithms produce stable score-limits (denoted by l_A and l_C).

Generalisation of the Gale-Shapley theorem

Observation: The applicant/college-oriented algorithms produce stable score-limits (denoted by l_A and l_C).

We say that $l \leq l'$ if $l(K) \leq l'(K)$ for every college K.

B. (2007): If l_* is a stable score-limit, then $l_A \leq l_* \leq l_C$.
(so l_A is the best and l_C is the worst for the applicants!)

Generalisation of the Gale-Shapley theorem

Observation: The applicant/college-oriented algorithms produce stable score-limits (denoted by l_A and l_C).

We say that $l \leq l'$ if $l(K) \leq l'(K)$ for every college K.

B. (2007): If l_* is a stable score-limit, then $l_A \leq l_* \leq l_C$.
(so l_A is the best and l_C is the worst for the applicants!)

Remark: the college-oriented algorithm was changed to the applicant-oriented version in 2007.

2nd special feature: LOWER QUOTAS

Each college C has lower quota, $l(C)$ and upper quota, $u(C)$.

A solution is a matching, where each college C has either
- no assignees ("closed college") or
- at least $l(C)$ and at most $u(C)$ assignees ("open college").

2nd special feature: LOWER QUOTAS

Each college C has lower quota, $l(C)$ and upper quota, $u(C)$.

A solution is a matching, where each college C has either
- no assignees ("closed college") or
- at least $l(C)$ and at most $u(C)$ assignees ("open college").

A matching is stable is there exist no
- "blocking pair", consisting of an open college and an unsatisfied applicant,
- "blocking coalition", consisting of a closed college C and $l(C)$ unsatisfied applicants.

CA-LQ: College Admission with lower quotas

An unsolvable instance, hardness

Studies:	**Saxophone**	**Trompet**
lower and upper quotas	$1 \leq \cdots \leq 1$	$2 \leq \cdots \leq 2$
1st applicant: 2nd applicant:	Adam Bill	Adam Bill

Adam's list: Trompet, Saxophone
Bill's list: Saxophone, Trompet

An unsolvable instance, hardness

Studies:	**Saxophone**	**Trompet**
lower and upper quotas	$1 \leq \cdots \leq 1$	$2 \leq \cdots \leq 2$
1st applicant: 2nd applicant:	Adam Bill	Adam Bill

Adam's list: Trompet, Saxophone
Bill's list: Saxophone, Trompet

An unsolvable instance, hardness

Studies:	**Saxophone**	**Trompet**
lower and upper quotas	$1 \leq \cdots \leq 1$	$2 \leq \cdots \leq 2$
1st applicant:	Adam	Adam
2nd applicant:	Bill	Bill

Adam's list: Trompet, Saxophone
Bill's list: Saxophone, Trompet

An unsolvable instance, hardness

Studies:	**Saxophone**	**Trompet**
lower and upper quotas	$1 \leq \cdots \leq 1$	$2 \leq \cdots \leq 2$
1st applicant: 2nd applicant:	Adam Bill	Adam Bill

Adam's list: Trompet, Saxophone
Bill's list: Saxophone, Trompet

An unsolvable instance, hardness

Studies:	**Saxophone**	**Trompet**
lower and upper quotas	$1 \leq \cdots \leq 1$	$2 \leq \cdots \leq 2$
1st applicant:	Adam	Adam
2nd applicant:	Bill	Bill

Adam's list: Trompet, Saxophone
Bill's list: Saxophone, Trompet

An unsolvable instance, hardness

Studies:	**Saxophone**	**Trompet**
lower and upper quotas	$1 \leq \cdots \leq 1$	$2 \leq \cdots \leq 2$
1st applicant: 2nd applicant:	Adam Bill	Adam Bill

Adam's list: Trompet, Saxophone
Bill's list: Saxophone, Trompet

An unsolvable instance, hardness

Studies:	**Saxophone**	**Trompet**
lower and upper quotas	$1 \leq \cdots \leq 1$	$2 \leq \cdots \leq 2$
1st applicant:	Adam	Adam
2nd applicant:	Bill	Bill

Adam's list: Trompet, Saxophone
Bill's list: Saxophone, Trompet

An unsolvable instance, hardness

Studies:	**Saxophone**	**Trompet**
lower and upper quotas	$1 \leq \cdots \leq 1$	$2 \leq \cdots \leq 2$
1st applicant:	Adam	Adam
2nd applicant:	Bill	Bill

Adam's list: Trompet, Saxophone
Bill's list: Saxophone, Trompet

B.-Fleiner-Irving-Manlove (2008): CA-LQ is NP-complete, even if no upper quota exceeds 3. (reduction from MAX-SMTI)

An unsolvable instance, hardness

Studies:	**Saxophone**	**Trompet**
lower and upper quotas	$1 \leq \cdots \leq 1$	$2 \leq \cdots \leq 2$
1st applicant:	Adam	Adam
2nd applicant:	Bill	Bill

Adam's list: Trompet, Saxophone
Bill's list: Saxophone, Trompet

B.-Fleiner-Irving-Manlove (2008): CA-LQ is NP-complete, even if no upper quota exceeds 3. (reduction from MAX-SMTI)

Open: if no lower quota exceeds 2.

Current heuristics

The most "unpopular" colleges are closed after each round according to their filling ratio...

Studies:	**Saxophone**	**Trompet**
lower and upper quotas	$2 \leq \cdots \leq 2$	$3 \leq \cdots \leq 3$
1st applicant: 2nd applicant:	Adam Bill	Bill Cliff

Adam's list: Saxophone
Bill's list: Trompet, Saxophone
Cliff's list: Trompet

Current heuristics

The most “unpopular” colleges are closed after each round according to their filling ratio...

Studies:	**Saxophone**	**Trompet**
lower and upper quotas	$2 \leq \cdots \leq 2$	$3 \leq \cdots \leq 3$
1st applicant:	Adam	Bill
2nd applicant:	Bill	Cliff

Adam’s list: Saxophone
Bill’s list: Trompet, Saxophone
Cliff’s list: Trompet

After the first round, the filling ratios are:
Saxophone: 1/2, Trompet: 2/3

Current heuristics

The most “unpopular” colleges are closed after each round according to their filling ratio...

Studies:		**Trompet**
lower and upper quotas		$3 \leq \cdots \leq 3$
1st applicant: 2nd applicant:		Bill Cliff

Adam’s list:
Bill’s list: Trompet,
Cliff’s list: Trompet

After the first round, the filling ratios are:
Saxophone: 1/2, Trompet: 2/3
so Saxophone becomes closed!

Current heuristics

The most “unpopular” colleges are closed after each round according to their filling ratio...

Studies:		**Trompet**
lower and upper quotas		$3 \leq \cdots \leq 3$
1st applicant:		Bill
2nd applicant:		Cliff

Adam’s list:
Bill’s list: Trompet,
Cliff’s list: Trompet

and after the next round we close Trompet as well...

Current heuristics

The most "unpopular" colleges are closed after each round according to their filling ratio...

Studies:	**Saxophone**	**Trompet**
lower and upper quotas	$2 \leq \cdots \leq 2$	$3 \leq \cdots \leq 3$
1st applicant:	Adam	Bill
2nd applicant:	Bill	Cliff

Adam's list: Saxophone
Bill's list: Trompet, Saxophone
Cliff's list: Trompet

and after the next round we close Trompet as well...
However, a stable matching exists!
So this heuristics may not find a stable solution, even if one exists.

Size versus stability: an example

Studies:	**Guitar**	**Saxophone**	**Trompet**	**Singer**
quotas	$10 \leq \dots$	$1 \leq \dots \leq 1$	$2 \leq \dots \leq 2$	$10 \leq \dots$
1st applicant:	Adam	Adam	Bill	Bill
2nd applicant:	Cliff		Cliff	Tina
3rd applicant:	Jimi			Amy
...	...			...
10st applicant:	Kurt			Maria

Adam's list: Saxophone, Guitar
Bill's list: Trompet, Singer
Cliff's list: Guitar, Trompet
Other guitarists' list: Guitar
Other singers' list: Singer

Size versus stability: an example

Studies:	**Guitar**	**Saxophone**	**Trompet**	**Singer**
quotas	$10 \leq \ldots$	$1 \leq \cdots \leq 1$	$2 \leq \cdots \leq 2$	$10 \leq \ldots$
1st applicant:	Adam	Adam	Bill	Bill
2nd applicant:	Cliff		Cliff	Tina
3rd applicant:	Jimi			Amy
...	...			...
10st applicant:	Kurt			Maria

Adam's list: Saxophone, Guitar
Bill's list: Trompet, Singer
Cliff's list: Guitar, Trompet
Other guitarists' list: Guitar
Other singers' list: Singer

This matching is stable, but has **small size**.

Size versus stability: an example

Studies:	**Guitar**	**Saxophone**	**Trompet**	**Singer**
quotas	$10 \leq \dots$	$1 \leq \dots \leq 1$	$2 \leq \dots \leq 2$	$10 \leq \dots$
1st applicant:	Adam	Adam	Bill	Bill
2nd applicant:	Cliff		Cliff	Tina
3rd applicant:	Jimi			Amy
...	...			...
10st applicant:	Kurt			Maria

Adam's list: Saxophone, Guitar
Bill's list: Trompet, Singer
Cliff's list: Guitar, Trompet
Other guitarists' list: Guitar
Other singers' list: Singer

This matching is **maximum size**

Size versus stability: an example

Studies:	**Guitar**	**Saxophone**	**Trompet**	**Singer**
quotas	$10 \leq \ldots$	$1 \leq \cdots \leq 1$	$2 \leq \cdots \leq 2$	$10 \leq \ldots$
1st applicant:	Adam	Adam	Bill	Bill
2nd applicant:	Cliff		Cliff	Tina
3rd applicant:	Jimi			Amy
...	...			...
10st applicant:	Kurt			Maria

Adam's list: Saxophone, Guitar
Bill's list: Trompet, Singer
Cliff's list: Guitar, Trompet
Other guitarists' list: Guitar
Other singers' list: Singer

This matching is **maximum size**
but there is a blocking coalition!

Size versus stability: an example

Studies:	**Guitar**	**Saxophone**	**Trompet**	**Singer**
quotas	$10 \leq \ldots$	$1 \leq \cdots \leq 1$	$2 \leq \cdots \leq 2$	$10 \leq \ldots$
1st applicant:	Adam	Adam	Bill	Bill
2nd applicant:	Cliff		Cliff	Tina
3rd applicant:	Jimi			Amy
...	...			...
10st applicant:	Kurt			Maria

Adam's list: Saxophone, Guitar
Bill's list: Trompet, Singer
Cliff's list: Guitar, Trompet
Other guitarists' list: Guitar
Other singers' list: Singer
Heuristics: after the first round "Trompet" is the most unpopular

Size versus stability: an example

Studies:	**Guitar**	**Saxophone**		**Singer**
quotas	$10 \leq \dots$	$1 \leq \dots \leq 1$		$10 \leq \dots$
1st applicant:	Adam	Adam		Bill
2nd applicant:	Cliff			Tina
3rd applicant:	Jimi			Amy
...	...			...
10st applicant:	Kurt			Maria

Adam's list: Saxophone, Guitar
Bill's list: , Singer
Cliff's list: Guitar,
Other guitarists' list: Guitar
Other singers' list: Singer
Heuristics: after the first round "Trompet" is the most unpopular so it becomes closed

Size versus stability: an example

Studies:	**Guitar**	**Saxophone**		**Singer**
quotas	$10 \leq \ldots$	$1 \leq \cdots \leq 1$		$10 \leq \ldots$
1st applicant:	Adam	Adam		Bill
2nd applicant:	Cliff			Tina
3rd applicant:	Jimi			Amy
...	...			...
10st applicant:	Kurt			Maria

Adam's list: Saxophone, Guitar
Bill's list: Trompet, Singer
Cliff's list: Guitar,
Other guitarists' list: Guitar
Other singers' list: Singer
Heuristics: after the first round "Trompet" is the most unpopular so it becomes closed, and we get this matching.

Size versus stability: an example

Studies:	**Guitar**	**Saxophone**	**Trompet**	**Singer**
quotas	$10 \leq \dots$	$1 \leq \dots \leq 1$	$2 \leq \dots \leq 2$	$10 \leq \dots$
1st applicant:	Adam	Adam	Bill	Bill
2nd applicant:	Cliff		Cliff	Tina
3rd applicant:	Jimi			Amy
...	...			...
10st applicant:	Kurt			Maria

Adam's list: Saxophone, Guitar
Bill's list: Trompet, Singer
Cliff's list: Guitar, Trompet
Other guitarists' list: Guitar
Other singers' list: Singer
Heuristics: after the first round "Trompet" is the most unpopular so it becomes closed, and we get this matching.
Although a blocking coalition exists.

Relaxed problems

If no closed college can be blocking then a pairwise stable solution always exists for the open colleges.

MAX-PS-CA-LQ: maximise the number of admitted applicants

Relaxed problems

If no closed college can be blocking then a pairwise stable solution always exists for the open colleges.

MAX-PS-CA-LQ: maximise the number of admitted applicants

MAX-POP-CA-LQ: if no "popular" college can be closed

Relaxed problems

If no closed college can be blocking then a pairwise stable solution always exists for the open colleges.

MAX-PS-CA-LQ: maximise the number of admitted applicants

MAX-POP-CA-LQ: if no "popular" college can be closed

B.-Fleiner-Irving-Manlove (2008): MAX-PS-CA-LQ and MAX-POP-CA-LQ are NP-complete, even if no quota exceeds 3.

Relaxed problems

If no closed college can be blocking then a pairwise stable solution always exists for the open colleges.
MAX-PS-CA-LQ: maximise the number of admitted applicants
MAX-POP-CA-LQ: if no "popular" college can be closed

B.-Fleiner-Irving-Manlove (2008): MAX-PS-CA-LQ and MAX-POP-CA-LQ are NP-complete, even if no quota exceeds 3.

The proof is like the one by Cornuéjouls: $\mathrm{X3C} \longrightarrow \{0,3\}$-factor
$S_i = \{x_{i,1}, x_{i,2}, x_{i,3}\} \longrightarrow C_i : a_{i,1}, a_{i,2}, a_{i,3}$ with $l(C_i) = u(C_i) = 3$

Relaxed problems

If no closed college can be blocking then a pairwise stable solution always exists for the open colleges.

MAX-PS-CA-LQ: maximise the number of admitted applicants

MAX-POP-CA-LQ: if no "popular" college can be closed

B.-Fleiner-Irving-Manlove (2008): MAX-PS-CA-LQ and MAX-POP-CA-LQ are NP-complete, even if no quota exceeds 3.

The proof is like the one by Cornuéjouls: X3C $\longrightarrow$ $\{0,3\}$-factor

$S_i = \{x_{i,1}, x_{i,2}, x_{i,3}\} \longrightarrow C_i : a_{i,1}, a_{i,2}, a_{i,3}$ with $l(C_i) = u(C_i) = 3$

Remark: The hardness of the $\{0,3\}$-factor problem for non-bipartite graphs was proven by Lovász in 1971!

Relaxed problems

If no closed college can be blocking then a pairwise stable solution always exists for the open colleges.

MAX-PS-CA-LQ: maximise the number of admitted applicants

MAX-POP-CA-LQ: if no "popular" college can be closed

B.-Fleiner-Irving-Manlove (2008): MAX-PS-CA-LQ and MAX-POP-CA-LQ are NP-complete, even if no quota exceeds 3.

The proof is like the one by Cornuéjouls: X3C$\longrightarrow$ $\{0,3\}$-factor
$S_i = \{x_{i,1}, x_{i,2}, x_{i,3}\} \longrightarrow C_i : a_{i,1}, a_{i,2}, a_{i,3}$ with $l(C_i) = u(C_i) = 3$

Remark: The hardness of the $\{0,3\}$-factor problem for non-bipartite graphs was proven by Lovász in 1971!

Open: MAX-PS-CA-LQ and MAX-POP-CA-LQ if no lower quota exceeds 2.

3rd special feature: COMMON QUOTAS

Studies:	**p. $\mathbf{CS}_{BUTE}$**	**s. $\mathbf{CS}_{BUTE}$**	...	**s. $\mathbf{CS}_{G.D.}$**	...
c. quotas:		CS national quota: $\leq$3000			
quotas:	$\leq$ 50	$\leq$ 450	...	$\leq$ 400	...
2004:	49 (78p)	474 (113p)	...	336 (74p)	...
2005:	51 (90p)	423 (126p)	...	369 (77p)	...
2006:	41 (80p)	443 (125p)	...	321 (78p)	...
2007:	51 (100p)	478 (120p)	...	246 (79p)	...

3rd special feature: COMMON QUOTAS

Studies:	**p. CS_{BUTE}**	**s. CS_{BUTE}**	...	**s. $CS_{G.D.}$**	...
c. quotas:		CS national quota: $\leq$3000			
quotas:	$\leq$ 50	$\leq$ 450	...	$\leq$ 400	...
2004:	49 (78p)	474 (113p)	...	336 (74p)	...
2005:	51 (90p)	423 (126p)	...	369 (77p)	...
2006:	41 (80p)	443 (125p)	...	321 (78p)	...
2007:	51 (100p)	478 (120p)	...	246 (79p)	...

Studies:	**p. CS_{BUTE}**	**s. CS_{BUTE}**	...	**s. $CS_{G.D.}$**	...
c. quotas:		CS national quota: $\leq$3000			
c. quotas:	faculty quota: $\leq$500		...	$\leq$ 400	...
2008:	8 (365p)	492 (366p)	...	165 (160p)	...

Formal definition

Some set of colleges may have a common quota.
(this quota cannot be exceeded in a matching...)

Formal definition

Some set of colleges may have a common quota.
(this quota cannot be exceeded in a matching...)

The stability of a matching:
If an applicant a_i is not matched to a college C, then
– either a_i is matched to a better college
– or C has filled its quota with better applicants than a_i
– or there is a set of colleges S such that $C \in S$ and S filled its quota with better applicants.

Nested set systems: nice properties

A set system $\mathcal{C}$ is nested if, for every pair of sets in $\mathcal{C}$ such that $S \cap S' \neq \emptyset$, either $S \subseteq S'$ or $S \supseteq S'$ holds.

Nested set systems: nice properties

A set system $\mathcal{C}$ is nested if, for every pair of sets in $\mathcal{C}$ such that $S \cap S' \neq \emptyset$, either $S \subseteq S'$ or $S \supseteq S'$ holds.

B.-Fleiner-Irving-Manlove (2008): For nested set systems, a generalised Gale-Shapley algorithm produces a stable matching.

Nested set systems: nice properties

A set system $\mathcal{C}$ is nested if, for every pair of sets in $\mathcal{C}$ such that $S \cap S' \neq \emptyset$, either $S \subseteq S'$ or $S \supseteq S'$ holds.

B.-Fleiner-Irving-Manlove (2008): For nested set systems, a generalised Gale-Shapley algorithm produces a stable matching.

B.-Fleiner-Irving-Manlove (2008): The set of admitted applicants is the same in the stable matchings for a given instance of CA-CQ ("Rural Hospitals theorem"). Moreover, the set of stable matchings form a lattice.

Nested set systems: nice properties

A set system $\mathcal{C}$ is nested if, for every pair of sets in $\mathcal{C}$ such that $S \cap S' \neq \emptyset$, either $S \subseteq S'$ or $S \supseteq S'$ holds.

B.-Fleiner-Irving-Manlove (2008): For nested set systems, a generalised Gale-Shapley algorithm produces a stable matching.

B.-Fleiner-Irving-Manlove (2008): The set of admitted applicants is the same in the stable matchings for a given instance of CA-CQ ("Rural Hospitals theorem"). Moreover, the set of stable matchings form a lattice.

This was the case in the application until 2007!

Non-nested set systems: unsolvable and hard

Studies:	**p. Sax$_A$**	**s. Sax$_A$**	**s. Sax$_B$**	**Trompet**
c. quotas:		Sax n. quota: ≤ 1		≤ 1
c. quotas:	faculty quota: ≤ 1			
1st appl.:			Cliff	Adam
2nd appl.:		Bill		Cliff
3rd appl.:	Adam			

Adam's list: p. Sax$_A$, Trompet
Bill's list: s. Sax$_A$
Cliff's list: Trompet, s. Sax$_B$

Non-nested set systems: unsolvable and hard

Studies:	**p. Sax_A**	**s. Sax_A**	**s. Sax_B**	**Trompet**
c. quotas:		Sax n. quota: ≤ 1		≤ 1
c. quotas:	faculty quota: ≤ 1			
1st appl.:			Cliff	Adam
2nd appl.:		Bill		Cliff
3rd appl.:	Adam			

Adam's list: p. Sax_A, Trompet
Bill's list: s. Sax_A
Cliff's list: Trompet, s. Sax_B

Adam cannot be unmatched, otherwise he blocks with Trompet...

Non-nested set systems: unsolvable and hard

Studies:	**p. Sax$_A$**	**s. Sax$_A$**	**s. Sax$_B$**	**Trompet**
c. quotas:		Sax n. quota: $\leq$1		≤ 1
c. quotas:	faculty quota: $\leq$1			
1st appl.:			Cliff	Adam
2nd appl.:		Bill		Cliff
3rd appl.:	Adam			

Adam's list: p. Sax$_A$, Trompet
Bill's list: s. Sax$_A$
Cliff's list: Trompet, s. Sax$_B$

Suppose that Adam is matched to p. Sax$_A$,

Non-nested set systems: unsolvable and hard

Studies:	**p. Sax$_A$**	**s. Sax$_A$**	**s. Sax$_B$**	**Trompet**
c. quotas:		Sax n. quota: ≤ 1		≤ 1
c. quotas:	faculty quota: ≤ 1			
1st appl.:			Cliff	Adam
2nd appl.:		Bill		Cliff
3rd appl.:	Adam			

Adam's list: p. Sax$_A$, Trompet
Bill's list: s. Sax$_A$
Cliff's list: Trompet, s. Sax$_B$

Suppose that Adam is matched to p. Sax$_A$, then Cliff must be matched to Trompet and Bill blocks with s. Sax$_A$

Non-nested set systems: unsolvable and hard

Studies:	**p. Sax$_A$**	**s. Sax$_A$**	**s. Sax$_B$**	**Trompet**
c. quotas:		Sax n. quota: ≤ 1		≤ 1
c. quotas:	faculty quota: ≤ 1			
1st appl.:			Cliff	Adam
2nd appl.:		Bill		Cliff
3rd appl.:	Adam			

Adam's list: p. Sax$_A$, Trompet
Bill's list: s. Sax$_A$
Cliff's list: Trompet, s. Sax$_B$

Suppose that Adam is matched to Trompet...

Non-nested set systems: unsolvable and hard

Studies:	**p. Sax$_A$**	**s. Sax$_A$**	**s. Sax$_B$**	**Trompet**
c. quotas:		Sax n. quota: ≤ 1		≤ 1
c. quotas:	faculty quota: ≤ 1			
1st appl.:			Cliff	Adam
2nd appl.:		Bill		Cliff
3rd appl.:	Adam			

Adam's list: p. Sax$_A$, Trompet
Bill's list: s. Sax$_A$
Cliff's list: Trompet, s. Sax$_B$

Suppose that Adam is matched to Trompet...
Cliff must be matched to s. Sax$_B$, otwerwise they block

Non-nested set systems: unsolvable and hard

Studies:	**p. Sax_A**	**s. Sax_A**	**s. Sax_B**	**Trompet**
c. quotas:		Sax n. quota: ≤ 1		≤ 1
c. quotas:	faculty quota: ≤ 1			
1st appl.:			Cliff	Adam
2nd appl.:		Bill		Cliff
3rd appl.:	Adam			

Adam's list: p. Sax_A, Trompet
Bill's list: s. Sax_A
Cliff's list: Trompet, s. Sax_B

Suppose that Adam is matched to Trompet...
Cliff must be matched to s. Sax_B, otwerwise they block
So Bill cannot be matched to s. Sax_A,

Non-nested set systems: unsolvable and hard

Studies:	**p. Sax_A**	**s. Sax_A**	**s. Sax_B**	**Trompet**
c. quotas:		Sax n. quota: ≤ 1		≤ 1
c. quotas:	faculty quota: ≤ 1			
1st appl.:			Cliff	Adam
2nd appl.:		Bill		Cliff
3rd appl.:	Adam			

Adam's list: p. Sax_A, Trompet
Bill's list: s. Sax_A
Cliff's list: Trompet, s. Sax_B

Suppose that Adam is matched to Trompet...
Cliff must be matched to s. Sax_B, otwerwise they block
So Bill cannot be matched to s. Sax_A,
Thus, Adam blocks with p. Sax_A.

Non-nested set systems: unsolvable and hard

Studies:	**p. Sax_A**	**s. Sax_A**	**s. Sax_B**	**Trompet**
c. quotas:		Sax n. quota: ≤ 1		≤ 1
c. quotas:	faculty quota: ≤ 1			
1st appl.:			Cliff	Adam
2nd appl.:		Bill		Cliff
3rd appl.:	Adam			

Adam's list: p. Sax_A, Trompet
Bill's list: s. Sax_A
Cliff's list: Trompet, s. Sax_B

B.-Fleiner-Irving-Manlove (2008): CA-CQ is NP-complete, even if every quota is 1 and no college is included more than 2 sets. (Reduction from MAX-SMTI)

Non-nested set systems: unsolvable and hard

Studies:	**p. Sax$_A$**	**s. Sax$_A$**	**s. Sax$_B$**	**Trompet**
c. quotas:		Sax n. quota: ≤ 1		≤ 1
c. quotas:	faculty quota: ≤ 1			
1st appl.:			Cliff	Adam
2nd appl.:		Bill		Cliff
3rd appl.:	Adam			

Adam's list: p. Sax$_A$, Trompet
Bill's list: s. Sax$_A$
Cliff's list: Trompet, s. Sax$_B$

B.-Fleiner-Irving-Manlove (2008): CA-CQ is NP-complete, even if every quota is 1 and no college is included more than 2 sets. (Reduction from MAX-SMTI)

This is the case in the application since 2007!

Thank you for your attention!
Further information at
www.optimalmatching.com